

On recoverability properties of fixed measurement matrices

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Abstract

The purpose of this paper is to extend a result by Donoho and Huo, Elad and Bruckstein, Gribnoval and Nielsen on sparse representations of signals in dictionaries to general matrices. We consider a general fixed measurement matrix, not necessarily a dictionary, and derive sufficient condition for having unique sparse representation of signals in this matrix. Currently, to the best of our knowledge, no such method exists. In particular, if matrix is a dictionary, our method is at least as good as the method proposed by Gribnoval and Nielsen.

1 Introduction

Given a data vector $\tilde{x} \in \mathbb{R}^n$, the linear measurements y_i of the data \tilde{x} consist of the inner products of \tilde{x} with a number of measurement vectors $a_i \in \mathbb{R}^n$, $i = 1, 2, \dots, m$, that is $y_i = \langle a_i, \tilde{x} \rangle$. In matrix form $\tilde{y} = A\tilde{x}$, where A is an $m \times n$ matrix, called the measurement or encoding matrix, that consists of a_i 's as its rows and m is the number of measurements.

If the number of measurements is less than the dimension of the data, that is, $m < n$, the linear system $A\tilde{x} = \tilde{y}$ is under-determined, and therefore

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has infinitely many solutions, which makes the recovery of \tilde{x} impossible. However if (a) the data vector \tilde{x} is sufficiently sparse and (b) the encoding matrix A contains a sufficient number of measurements and satisfies certain properties, then \tilde{x} can be recovered (exactly or to a given accuracy) at a polynomial time complexity.

We consider the following recovery problem of a sparse vector $\tilde{x} \in \mathbb{R}^n$ from its linear measurement $\tilde{y} = A\tilde{x} \in \mathbb{R}^m$, where A is a known $m \times n$ full rank matrix and $m < n$. The associated optimization problem could be stated as

$$\min_{x \in \mathbb{R}^n} \{\|x\|_0 : Ax = \tilde{y}\}, \quad (1)$$

where $\|x\|_0$ is the number of nonzero entries of x . This problem is non-convex and therefore can not be solved by conventional optimization methods.

On the other hand we can solve the following problem which can be written as a linear program (LP) via a standard transformation,

$$\min_{x \in \mathbb{R}^n} \{\|x\|_1 : Ax = \tilde{y}\} \quad (2)$$

and ask a question: Under what conditions on A and \tilde{x} are the problems (1) and (2) uniquely solved by \tilde{x} ?

Definition 1 (Partition). *By a partition (S, Z) we mean a partition of the index set $\{1, 2, \dots, n\}$ into two disjoint subsets S and Z such that $S \cup Z = \{1, 2, \dots, n\}$ and $S \cap Z = \emptyset$. In particular, for any $x \in \mathbb{R}^n$, the partition $(S(x), Z(x))$ refers to the support $S(x)$ of x and its complement – the zero set $Z(x)$, namely*

$$S(x) = \{i : x_i \neq 0, 1 \leq i \leq n\}, \quad Z(x) = \{i : x_i = 0, 1 \leq i \leq n\}. \quad (3)$$

Definition 2 (k -balancedness). *A subspace $V \subseteq \mathbb{R}^n$ is k -balanced (in l_1 norm) if for any partition (S, Z) with cardinality of S equals to k*

$$\|v_S\|_1 \leq \|v_Z\|_1, \forall v \in V.$$

It is strictly k -balanced if the strict inequality holds for all $v \neq 0$.

Definitions of k -balancedness was introduced by Zhang in [7]. However, k -balancedness was used by Donoho and Huo in [2], Elad and Bruckstein in [4], Gribnoval and Nielsen in [5].

Theorem 1 (Necessary and Sufficient Conditions for Recovery). *Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times n}$ be full rank such that $p + m = n$ and $AB^T = 0$. Then for any \tilde{x} with $\|\tilde{x}\|_0 \leq k$ and $\tilde{y} = A\tilde{x}$, \tilde{x} uniquely solves (1) and (2) if and only if $\text{range}(B^T) \subset \mathbb{R}^n$ is strictly k -balanced.*

In [7] Zhang stated Theorem 1 in its current form and gave a simple proof by connecting equivalent recoverability conditions for different spaces. The theorem was used without being stated explicitly by Donoho and Huo in [2] and by Elad and Bruckstein in [4] and was stated as Lemma by Gribnoval and Nielsen in [5].

Definition 3 (Dictionary). *We say that A is a dictionary if the columns of A are unit vectors.*

Definition 4 (Coherence of a Dictionary). *Let $A \in \mathbb{R}^{m \times n}$ be a dictionary. The coherence of a dictionary $M(A)$ is defined by*

$$M(A) = \max_{i \neq j} |\langle a_i, a_j \rangle|, \quad (4)$$

where $a_i, 1 \leq i \leq n$, is the i -th column of A .

Next theorem is due to Gribnoval and Nielsen [5].

Theorem 2. *Let k be a natural number and let $\|\tilde{x}\|_0 \leq k$. For any dictionary A , if $k < \frac{1}{2} \left(1 + \frac{1}{M(A)}\right)$ and $\tilde{y} = A\tilde{x}$, then \tilde{x} is the unique solution to both (1) and (2).*

It is necessary to mention that if m is a power of 2, then there exists a dictionary A such that $M(A) = \frac{1}{\sqrt{m}}$. See, for example [1] and [6].

2 Main Result

Definition 5 ($\gamma_{1,\infty}$ -width). *Let both $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times n}$ be of full rank and $p + m = n$ and $AB^T = 0$. We define $\gamma_{1,\infty}$ -width of A to be*

$$\gamma_{1,\infty}(A) = \min_{x \in \mathbb{R}^p, x \neq 0} \frac{\|B^T x\|_1}{\|B^T x\|_\infty} = \min_{\|B^T x\|_\infty = 1} \|B^T x\|_1. \quad (5)$$

The feasible set $\{x \in \mathbb{R}^p : \|B^T x\|_\infty = 1\}$ is non-convex, however, it is a union of $2n$ convex sets

$$\{x \in \mathbb{R}^p : \|B^T x\|_\infty = 1\} = \bigcup_{i=1}^n \pm F_i, \quad (6)$$

where

$$F_i = \{x \in \mathbb{R}^p : [B^T x]_i = 1; |[B^T x]_j| \leq 1, j \neq i\}. \quad (7)$$

Therefore,

$$\gamma_{1,\infty}(A) = \min_{1 \leq i \leq n} \min_{x \in F_i} \|B^T x\|_1. \quad (8)$$

For every $1 \leq i \leq n$, $\min_{x \in F_i} \|B^T x\|_1$ could be rewritten as a linear program via a standard transformation. Therefore, in order to compute $\gamma_{1,\infty}(A)$ it is necessary to solve n linear programs. While it requires considerable computational efforts for a large n , the problem is solvable in polynomial time.

Alternatively, one can solve the reciprocal problem

$$\frac{1}{\gamma_{1,\infty}(A)} = \max_{\|B^T x\|_1 = 1} \|B^T x\|_\infty. \quad (9)$$

Next proposition presents sufficient condition for recovery. It follows directly from Theorem 1.

Proposition 1 (Sufficient Condition for Recovery). *Recovery is guaranteed whenever $k < \frac{1}{2}\gamma_{1,\infty}(A)$.*

Proof. Note, that $\|v_S\|_1 \leq \|v_Z\|_1$ is equivalent to $\|v_S\|_1 \leq \frac{1}{2}\|v\|_1$. Therefore,

$$\|v_S\|_1 \leq k\|v_S\|_\infty \leq k\|v\|_\infty < \frac{1}{2}\gamma_{1,\infty}(A)\|v\|_\infty \leq \frac{1}{2} \frac{\|v\|_1}{\|v\|_\infty} \|v\|_\infty \leq \frac{1}{2}\|v\|_1.$$

□

Now we are ready to show that estimated sparsity k for guaranteed recovery of a dictionary A computed using $\gamma_{1,\infty}$ -width of A is always greater or equal to the estimated sparsity k computed using coherence of A .

Theorem 3. *Let $A \in \mathbb{R}^{m \times n}$ be a dictionary and $m < n$. Let $B \in \mathbb{R}^{p \times n}$, such that $p + m = n$ and $AB^T = 0$. Let k_1 and k_2 be the sparsities for guaranteed recovery estimated by $\gamma_{1,\infty}(A)$ and $M(A)$ respectively. Then $k_1 \geq k_2$.*

Proof. According to Theorem 2 and Proposition 1, it is enough to show that

$$1 + \frac{1}{M(A)} \leq \gamma_{1,\infty}(A). \quad (10)$$

We will follow the proof of Gribnoval and Nielsen [5].

Let $v \in \text{range}(B^T)$, then $Av = 0$, or, in vector form $\sum_{i=1}^n v_i a_i = 0$, where a_i , $1 \leq i \leq n$ is the i -th column of A . Then, $v_1 a_1 = -\sum_{i=2}^n v_i a_i$. Taking the inner product of both sides with a_1 , we get $v_1 = -\sum_{i=2}^n v_i \langle a_i, a_1 \rangle$. It follows that

$$|v_1| = \left| -\sum_{i=2}^n v_i \langle a_i, a_1 \rangle \right| \leq M(A) \sum_{i=2}^n |v_i| = M(A)(\|v\|_1 - |v_1|), \quad (11)$$

or

$$|v_1|(1 + M(A)) \leq \|v\|_1 M(A). \quad (12)$$

The same way for $2 \leq i \leq n$, we get

$$|v_i|(1 + M(A)) \leq \|v\|_1 M(A). \quad (13)$$

Since this is true for every index $1 \leq i \leq n$, it follows that for every vector $v \in \text{range}(B^T)$ the following inequality holds:

$$1 + \frac{1}{M(A)} \leq \frac{\|v\|_1}{\|v\|_\infty}. \quad (14)$$

Now if we take minimum over all $v \in \text{range}(B^T)$ we get:

$$1 + \frac{1}{M(A)} \leq \min_v \frac{\|v\|_1}{\|v\|_\infty} = \gamma_{1,\infty}(A). \quad (15)$$

which completes the proof. \square

3 Conclusion

In this paper we defined $\gamma_{1,\infty}$ -width of a measurement matrix A and showed that if A is a dictionary, our approach to estimate recoverability properties of A is at least as good as coherence approach. Moreover, our method can be used to estimate the recoverability of A even in the case A is not a dictionary. Currently, to the best of our knowledge, no other such method exists.

References

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